Tense SHn–algebras

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Abstract

In 1982, L. Iturrioz introduced symmetrical Heyting algebras of order n (or SHn–algebras). In this paper, we define and study tense SHn–algebras namely, SHn–algebras endowed with two tense operators. These algebras constitute a generalization of tense Łukasiewicz–Moisil algebras. Our main interest is the duality theory for tense SHn–algebras. In order to do this, we requiere Esakia's duality for Heyting algebras and Goldblatt's duality for bounded distributive lattices with operations.

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1 Introduction

Classical tense logic is a logical system obtained from bivalent logic by adding the tense operators G (it is always going to be the case that) and H (it has always been the case that)(see [1]). Taking into account that tense algebras constitute the algebraic basis for the bivalent tense logic (see [1, 9]), D. Diaconescu

[1]G. Pelaitay dedicates this work to his wife, Marianela on the occasion of his 24th birthday.

and G. Georgescu introduced in [2] tense MV–algebras and tense Łukasiewicz–Moisil algebras as algebraic structures for some many–valued tense logics. Starting with other logical systems and adding appropiate tense operators, we produce new tense logics.

On the other hand, it has been showed that Post and Lukasiewicz–Moisil algebras are both Heyting algebras with operators. In both Łukasiewicz–Moisil and Post algebras a symmetry can be expressed in terms of the primitive operations. This led to the study of more general algebras, called *symmetrical Heyting* algebras of order n (or SHn–algebras) (see [6, 7, 8, 12]). In the present paper we define and study tense SHn–algebras, namely, SHn–algebras endowed with two tense operators. These algebras constitute a generalization of tense Łukasiewicz–Moisil algebras and thus, they can offer an algebraic framework in order to develop some tense propositional SHn–logics.

2 Preliminaries

In this section, in order to simplify reading, we summarize the fundamental concepts we use.

Recall that Sofronie–Stokkermas [12] introduce the category **SHn** of SHn–algebras and SHn–homomorphisms, where a SHn–algebra is an algebra $\langle A, \vee, \wedge, \to, \sim, 0, 1, S_1, \ldots, S_{n-1}\rangle$ such that the reduct $\langle A, \vee, \wedge, \to, \sim, 0, 1\rangle$ is a symmetric Heyting algebra (see [10, 11]) and $S_1, \ldots, S_{n-1}$ are unary operators defined on A fulfilling the following equalities:

(S1) $S_i(x \wedge y) = S_i(x) \wedge S_i(y)$.

(S2) $S_i(x \to y) = \bigwedge_{k=i}^{n}(S_k(x) \to S_k(y))$.

(S3) $S_i(S_j(x)) = S_j(x)$, for every $i, j = 1, \ldots, n-1$.

(S4) $S_1(x) \vee x = x$.

(S5) $S_i(\sim x) = \sim S_{n-i}(x)$, for $i = 1, \ldots, n-1$.

(S6) $S_1(x) \vee \neg S_1(x) = 1$, with $\neg x = x \to 0$.

In addition, this author extended Esakia duality [3, 4] to the category **SHnSp** whose objects are SHn–spaces and whose morphisms are SHn–functions.

Specifically, a SHn–space is a system $(X, \leq, \tau, s_1, \ldots, s_{n-1}, g)$ such that $(X, \leq, \tau)$ is an Esakia space, $s_1, \ldots, s_{n-1}, g$ are continuous and for every $x, y \in X$ the following conditions are satisfied:

(E1) If $x \leq y$, then $g(y) \leq g(x)$.

(E2) $g(s_i(x)) = s_{n-i}(g(x))$.

(E3) $g(g(x)) = x$.

(E4) $s_j(s_i(x)) = s_j(x)$.

(E5) $s_1(x) \leq x$.

(E6) $x \leq s_{n-1}(x)$.

(E7) $s_i(x) \leq s_j(x)$ for every $i \leq j$.

(E8) If $x \leq y$, then $s_i(x) = s_i(y)$.

(E9) For all $i = 1, \dots, n-1$, $x \leq s_i(x)$ or $s_{i+1}(x) \leq x$.

and a SHn–function from a SHn–space $(X, \leq, \tau, s_1, \dots, s_{n-1}, g)$ into another $(X', \leq', \tau', s'_1, \dots, s'_{n-1}, g')$ is an Esakia morphism (continuos bounded morphism) $f : X \to X'$ such that preserve the operations $s_1, \dots, s_{n-1}, g$. Besides, he proved that **SHn** is dually equivalent to **SHnSp**.

On the other hand, in [5] R. Goldblatt obtained a topological duality for bounded distributive lattices with operators, i.e. with a family of join–hemimorphisms and/or meet–hemimorphisms. Now, we will describe this duality in the particular case of bounded distributive lattices endowed with two unary meet–hemimorphisms, G, H, which from now on will be called O–lattices.

A gP–space is a triple (X, R_G, R_H) where X is a compact totally order–disconnected topological space (or P–space), $R_G, R_H \subseteq X \times X$ are decreasing and the following conditions are satisfied:

(R1) For every $x \in X$, $R_G^{-1}(x)$ and $R_H^{-1}(x)$ are closed subsets of X.

(R2) For each $U \in D(X)$, $G_{R_G}(U), H_{R_H}(U) \in D(X)$, where

$G_{R_G}(U) = \{y \in X : R_G^{-1}(y) \subseteq U\}$, $H_{R_H}(U) = \{y \in X : R_H^{-1}(y) \subseteq U\}$ and $D(X)$ is the set of all clopen increasing subsets of X.

A gP–function from a gP–space (X_1, R_{G_1}, R_{H_1}) into another one, (X_2, R_{G_2}, R_{H_2}), is a continuous order–preserving function (or P–function) $f : X_1 \to X_2$ which satisfies the following conditions:

(r1) $(x, y) \in R_{G_1}$ implies $(f(x), f(y)) \in R_{G_2}$ for $x, y \in X_1$.

(r2) $(y, f(z)) \in R_{G_2}$ implies that there is $x \in X_1$ such that $(x, z) \in R_{G_1}$ and $f(x) \leq y$ for $z \in X_1$ and $y \in X_2$.

(r3) $(x, y) \in R_{H_1}$ implies $(f(x), f(y)) \in R_{H_2}$ for $x, y \in X_1$.

(r4) $(y, f(z)) \in R_{H_2}$ implies that there is $x \in X_1$ such that $(x, z) \in R_{H_1}$ and $f(x) \leq y$ for $z \in X_1$ and $y \in X_2$.

In [5] it was shown that

(G1) if (A, G, H) is an O–lattice and $R^A_T \subseteq X(A) \times X(A)$ is defined by $R^A_T = \{(P, F) \in X(A) \times X(A) : T^{-1}(F) \subseteq P\}$ for $T = G$ and $T = H$, then $(X(A), R^A_G, R^A_H)$ is a gP–space, where $X(A)$ is the set of all prime filters of A.

(G2) if (X, R_G, R_H) is a gP–space, then $(D(X), G_{R_G}, H_{R_H})$ is an O–lattice, where G_{R_G} and H_{R_H} are defined in (R2).

Taking into account (G1) and (G2) it is proved that the category of gP–spaces and gP–functions is dually equivalent to the category of O–lattices and their corresponding homomorphisms.

3 Duality for Tense SHn–algebras

In this section, we will study SHn–algebras endowed with two tense operators, more precisely:

Definition 3.1 *A tense SHn–algebra is an algebra $\langle A, \vee, \wedge, \rightarrow, \sim, S_1, \ldots, S_{n-1}, G, H, 0, 1\rangle$, where the reduct $\langle A, \vee, \wedge, \rightarrow, \sim, S_1, \ldots, S_{n-1}, 0, 1\rangle$ is a SHn–algebra and G, H are unary operators on A verifying the following conditions:*

(T1) $G(1) = 1,\ H(1) = 1,$

(T2) $G(x \wedge y) = G(x) \wedge G(y),\ H(x \wedge y) = H(x) \wedge H(y),$

(T3) $x \leq G(\sim H(\sim x)),\ x \leq H(\sim G(\sim x)),$

(T4) $S_i(G(x)) = G(S_i(x)),\ S_i(H(x)) = H(S_i(x))$, *for* $i = 1, \ldots, n-1$.

By **tSHn**, we will denote the category of tense SHn–algebras and their corresponding homomorphims.

Example 3.2 *If $\langle A, \vee, \wedge, \rightarrow, \sim, S_1, \ldots, S_{n-1}, G, H, 0, 1\rangle$ is a tense SHn–algebra in which satisfies the identity $(x \wedge \sim x) \vee (y \vee \sim y) = y \vee \sim y$, then $\langle A, \vee, \wedge, \sim, S_1, \ldots, S_{n-1}, G, H, 0, 1\rangle$ is a tense Łukasiewicz–Moisil algebra (see [6, 2]).*

We will indicate a Priestley–style duality for tense SHn–algebras

Definition 3.3 *A t–space is a system $(X, \leq, \tau, s_1, \ldots, s_{n-1}, g, R_G, R_H)$ where $(X, \leq, \tau, s_1, \ldots, s_{n-1}, g)$ is a SHn–space, (X, R_G, R_H) is a gP–space and the following additional conditions are satisfied:*

(t1) $(x, y) \in R_T$ *implies* $(s_i(x), s_i(y)) \in R_T$ *for* $T = G$ *and* $T = H$.

(t2) $(y, s_i(z)) \in R_T$ *implies that there is* $x \in X$ *such that* $(x, z) \in R_T$ *and* $s_i(x) \leq y$ *for* $T = G$ *and* $T = H$.

(t3) $(g(x), y) \in R_G$ *if and only if* $(g(y), x) \in R_H$.

Let **tSp** be the category whose objects are t–spaces and whose morphisms are SHn–functions which are also gP–functions.

For every t–space $(X, \leq, \tau, s_1, \ldots, s_{n-1}, g, R_G, R_H)$ let $D(X)$ be the set of all clopen increasing subsets of X. On $D(X)$ the following operations can be defined:

(a) $\vee : D(X) \times D(X) \to D(X)$ is defined by $U \vee V := U \cup V$,

(b) $\wedge : D(X) \times D(X) \to D(X)$ is defined by $U \wedge V := U \cap V$,

(c) $\to: D(X) \times D(X) \to D(X)$ is defined by $U \to V := \{x \in X : x \leq y \,\text{and}\, y \in U \,\text{implies}\, y \in V\}$,

(d) $\sim: D(X) \to D(X)$ is defined for every $U \in D(X)$ by $\sim U := X \setminus g(U)$,

(e) For every $i = 1, \ldots, n-1$, $S_i : D(X) \to D(X)$ is defined for every $U \in D(X)$ by $S_i(U) := s_i^{-1}(U)$.

For every morphism of t–space $\varphi : X \to X'$ let $D(\varphi) : D(X') \to D(X)$ be defined by $D(\varphi)(U) := \varphi^{-1}(U)$ for every $U \in D(X')$.

Lemma 3.4 *The functor* D : **tSP** $\to$ **tSHn** *is well–defined, i.e., the following holds:*

(i) The algebra $(D(X), \vee, \wedge, \to, \sim, S_1, \ldots, S_{n-1}, G_{R_G}, H_{R_H}, \emptyset, X)$ *is a tense* SHn*–algebra.*

(ii) If $\varphi : X \to X'$ *is a morphism of* t*–spaces, then* $D(\varphi) : D(X') \to D(X)$ *is a morphism of tense* SHn*–algebras.*

Proof We shall only prove that for all $U \in D(X)$, (T3) and (T4) are satisfied.

(T3) Let $y \in U$ and $(z, y) \in R_G$. Suppose that $z \in g(H_{R_H}(\sim U))$. Then $z = g(x)$ for some $x \in H_{R_H}(\sim U)$. From this last assertion, we infer that $R_H^{-1}(x) \subseteq \sim U$. Besides, since $g(x) \in R_G^{-1}(y)$, by (t3), we obtain that $g(y) \in R_H^{-1}(x)$ and so, $g(y) \in \sim U$. Hence, $g(y) \notin g(U)$, which contradicts that $y \in U$. Thus, $z \in \sim H_{R_H}(\sim U)$, from which we conclude that $R_G^{-1}(y) \subseteq \sim H_{R_H}(\sim U)$. So, $U \subseteq G_{R_G}(\sim H_{R_H}(\sim U))$. Similarly, it is proved that $U \subseteq H_{R_H}(\sim G_{R_G}(\sim U))$.

(T4) Let $s_i(y) \in G_{R_G}(U)$ and $(x,y) \in R_G$. Then, from (t1) we have that $(s_i(x), s_i(y)) \in R_G$. From this last assertion and (R2) we infer that $s_i(x) \in U$. Therefore, $S_i(G_{R_G}(U)) \subseteq G_{R_G}(S_i(U))$. On the other hand, let $z \in G_{R_G}(S_i(U))$ and $(y, s_i(z)) \in R_G$. Then by virtue of (t2), there is $x \in X$ such that $(x,z) \in R_G$ and $s_i(x) \leq y$. This last assertion allows us to conclude that $y \in U$. Similarly, it is proved that $S_i(H_{R_H}(U)) = H_{R_H}(S_i(U))$. This completes the proof.

For every tense SHn–algebra $\langle A, \vee, \wedge, \rightarrow, \sim, S_1, \ldots, S_{n-1}, G, H, 0, 1\rangle$ let $\mathbf{X(A)} = (X(A), \subseteq, \tau, s_1, ..., s_{n-1}, g, R_G^A, R_H^A)$, where $(X(A), \subseteq, \tau, s_1, ..., s_{n-1}, g)$ is the Priestley space of the SHn–algebra A (see [12]) and R_G^A, R_H^A are defined in (G1).

For every morphism of tense SHn–algebras $f : A \rightarrow A'$ let $X(f) : X(A') \rightarrow X(A)$ be defined by $X(f)(F) := f^{-1}(F)$ for every $F \in X(A')$.

Lemma 3.5 *The functor X : $\mathbf{tSHn} \rightarrow \mathbf{tSp}$ is well–defined, i.e., the following holds:*

(i) For every tense SHn–algebra $\langle A, \vee, \wedge, \rightarrow, \sim, S_1, \ldots, S_{n-1}, G, H, 0, 1\rangle$, $\mathbf{X(A)}$ is a t–space.

(ii) If $f : A \rightarrow A'$ is a morphism of tense SHn–algebras, then $X(f) : X(A') \rightarrow X(A)$ is a morphism of t–spaces.

Proof We shall only prove that for all $F, P \in X(A)$, (t1),(t2) and (t3) are satisfied.

(t1) It is a direct consequence of (G1) and (T4).

(t2) Let $G^{-1}(s_i(F)) \subseteq P$ and considering $E = \{z \in A : G(z) \in F\}$. Then we have that $\bigwedge I \not\leq \bigvee J$, for all finite subsets $I \subseteq E$, $J \subseteq S_i(A \setminus P)$. Indeed: Suppose that there is $I \subseteq E$, $J \subseteq S_i(A \setminus P)$ finite subsets such that $\bigwedge I \leq \bigvee J$. From this last assertion and (T2) we infer that $G(\bigwedge I) \in F$. Since, G is increasing we obtain that $G(\bigvee J) \in F$. On the other hand, it is straightforward to prove that $\bigvee J \in S_i(A \setminus P)$. From this last assertion, there is $a \in A \setminus P$ such that $S_i(a) = \bigvee J$. Hence, $G(S_i(a)) \in F$. Then, from (T4) we have deduced that $a \in P$, which is a contradiction. Therefore E is separated (see [5, page. 185]) from $S_i(A \setminus P)$, then from [5, page. 186], there is $Z \in X(A)$ such that $E \subseteq Z$ and $Z \cap S_i(A \setminus P) = \emptyset$. This last assertion allows us to conclude that $s_i(Z) \subseteq P$ and $(Z, F) \in R_G^A$. Similarly, it is proved (t2) for $T = H$.

(t3) Let $a \in H^{-1}(P)$ and suppose that $a \notin g(F)$. Then $\sim a \in F$. Besides, from (T3) we have that $\sim a \leq G(\sim H(a))$. So, $G(\sim H(a)) \in F$. On the other hand, from the hypothesis we infer that $G^{-1}(F) \subseteq g(P)$. Hence,

$\sim H(a) \in g(P)$. This last assertion allows us to conclude that $H(a) \notin P$ which is a contradiction. Therefore, $a \in g(F)$ and so, $g(F) \in (R_H^A)^{-1}(P)$. The other implication is proved in a similar way.

From Lemma 3.4 and 3.5 and taking into account the results indicated in [5, 12] we have

Theorem 3.6 *The categories* **tSHn** *and* **tSp** *are dually equivalent.*

References

[1] J. Burgess, *Basic tense logic*, Handbook of Philosophical Logic, vol. II, 89–133, Synthese Lib. 165, Reidel, Dordrecht, 1984.

[2] D. Diaconescu and G. Georgescu, *Tense operators on MV-algebras and Łukasiewicz-Moisil algebras*, Fund. Inform. **81** (2007), no. 4, 379–408.

[3] L. L. Esakia, *Topological Kripke models*, Soviet Math. Dokl. **15** (1974), 147–151.

[4] L. L. Esakia, *The problem of dualism in the intuitionistic logic and Browerian lattices*, In V Inter. Congress of Logic, Methodology and Philosophy of Science, pages 7–8. Canada, 1975.

[5] R. Goldblatt, *Varieties of complex algebras*, Ann. Pure Appl. Logic **44** (1989), no. 3, 173–242.

[6] L. Iturrioz, *Modal Operators on Symmetrical Heyting algebras*, Universal Algebra and Applications, Banach Center Publications 9, Traczyk T. (ed.), PWN-Polish Scientific Publishers (1982), 289–303.

[7] L. Iturrioz, *Symmetrical Heyting algebras with operators*, Z. Math. Logik Grundlag. Math. **29** (1983), no. 1, 33–70.

[8] L. Iturrioz and V. Sofronie–Stokkermans, *SHn–algebras (abbreviation of Symmetrical Heyting algebras of order n)*, chapter 4 in Atlas of Many–Valued Structures, COST Action 15, L. Iturrioz, E. Or lowska, E. Turunen (eds.), Tampere Univ. of Technology, Tampere, Finland, 2000, 11 pages, ISSN 1235-9599.

[9] T. Kowalski, *Varieties of tense algebras*, Rep. Math. Logic. **32** (1998), 53–95.

[10] A. Monteiro, *Sur les algèbres de Heyting Simetriques*, Special issue in honor of António Monteiro. Portugal. Math. **39** (1980), no. 1–4, 1–237 .

[11] H.P. Sankappanavar, *Heyting algebras with a dual lattice endomorphism*, Z. Math. Logik Grundlag. Math. **33** (1987), no. 6, 565–573.

[12] V. Sofronie-Stokkermans, *Priestley Duality for SHn-Algebras and Applications to the Study of Kripke–Style Models for SHn–Logics*, Multiple Valued Logic, An International Journal, 1999.